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Stress distribution in elastic isotropic semi-space with concentrated vertical force

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The distribution of stresses in elastic isotropic semi-space in horizontal and vertical direction under the effect of concentrated vertical force is investigated. A transition to line influences for stresses and their determination to arbitrary load is performed. Analysis and comparison of the results obtained is made.

Keywords: Elastic isotropic semi-space, Stress distribution, Line influences

INTRODUCTION

Elastic halfplane is a disk limited only on one straightlinear end and spread to infinity on one side of this end. Such is the stress and deformation state of a disk, loaded on its contour, the dimensions of which are too big in comparison with the length of the loading part. The solution of infinite elastic halfplane at uniformly distributed load (problem of Boussinesq), with concentrated load (problem of Flammant) and at arbitrary distributed load is discussed in [1]. Expressions are derived for the stresses at an arbitrary point of the halfplane. Through limits transition of these problems the expressions for determination of stresses under the effect of concentrated forces are obtained. A similar problem is encountered in the investigation of beams of elastic foundation, long strips, fundaments, etc.

The purpose of the present work is to determine the distribution of the stresses in the elastic halfplane in horizontal and vertical direction under the effect of concentrated force, to construct the influences for the stresses and through the obtained influences to determine the stresses at arbitrary load.

STRESSES IN THE HALFPLANE

Normal and tangential stresses are known from strength of materials. In the case of concentrated vertical force acting on the top edge, the expressions of the stresses in an arbitrary point of the halfplane are as presented in Figure1:

where

x, y are the coordinates of a point at which the stresses should be determined;

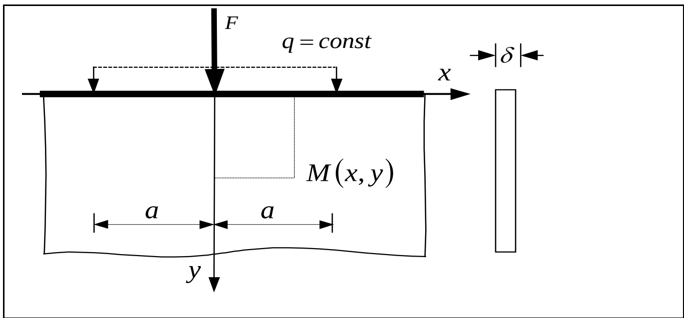
q is the intensity of load equivalent to the force F, distributed uniformly on the part with length $2a$ and symmetrically located about axis y;

The introduced coordinate system xOy is in the middle of the distributed load (Fig.1).

As a general signification of stresses $S(x,y)$ is used.

The written expressions are applied for investigation of the elastic halfplane loaded with concentrated force, with the characteristics $a = 10\,m$; $F = 1\frac{kN}{m}$. The intensity of the load in the cases is to coordinate with a single value of the force.

The analysis of the distribution of stresses is made with arbitrary and different in this case and in particular, multiple of a, an accepted discretization step in the axe's directions of the halfplane. Calculations are made with compound programs of PC.



$$\sigma_x = -\frac{2F}{\pi}\frac{x^2 y}{\left(x^2+y^2\right)^2},$$

$$\sigma_y = -\frac{2F}{\pi}\frac{y^3}{\left(x^2+y^2\right)^2},$$

$$\tau_{xy} = -\frac{2F}{\pi}\frac{xy^2}{\left(x^2+y^2\right)^2}.$$

Fig. 1. The expressions of the stresses in an arbitrary point of the halfplane

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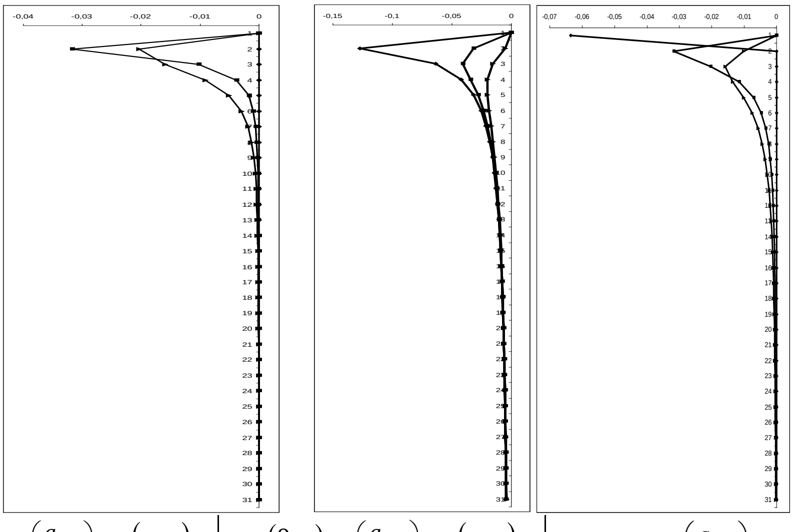
Stresses distribution in vertical direction

The solution of the stresses is made at $0 \le y \le 15a$ by step $\frac{a}{2}$. The results of the solutions are shown in Table 1.

Table 1. The solution of the stresses is made at 0≤y≤15a by step a/2

y	$\sigma_x(0,y)$	$\sigma_x\left(\frac{a}{2},y\right)$	$\sigma_x(a,y)$	$\sigma_y(0,y)$	$\sigma_y\left(\frac{a}{2},y\right)$	$\sigma_y(a,y)$	$\tau_{xy}(0,y)$	$\tau_{xy}\left(\frac{a}{2},y\right)$	$\tau_{xy}(a,y)$
0	-6,3649	-2,5465e-7	-6,3662e-8	-6,3649e-4	0,0000	0,0000	-0,0636	-5,0930e-13	-6,3662e-14
5	-5,0930e-9	-0,0318	-0,0204	-0,1273	-0,0318	-5,0930e-3	-2,5465e-5	-0,0318	-0,0102
10	-6,3662e-10	-0,0102	-0,0159	-0,0637	-0,0407	-0,0159	-6,3662e-6	-0,0204	-0,0159
15	-1,8863e-10	-3,8197e-3	-9,0408e-3	-0,0424	-0,0344	-0,0203	-2,8294e-6	-0,0115	-0,0136
20	-7,9578e-11	1,7623e-3	-5,0930e-3	-0,0318	-0,0282	-0,0204	-1,5916e-6	-7,0491e-3	-0,0102
25	-4,0744e-11	-9,4175e-4	-3,0279e-3	-0,0255	-0,0235	-0,0189	-1,0186e-6	-4,7087e-3	-7,5698e-3
30	-2,3579e-11	-5,5803e-4	-1,9099e-3	-0,0212	-0,0201	-0,0172	-7,0736e-7	-3,3482e-3	-5,7296e-3
35	-1,4848e-11	-3,5651e-4	-1,2692e-3	-0,0182	-0,0175	-0,0155	-5,1969e-7	-2,4956e-3	-4,4421e-3
40	-9,9472e-12	-2,4109e-4	-8,8114e-4	-0,0159	-0,0154	-0,0141	-3,9789e-7	-1,9287e-3	-3,5245e-3
45	-6,9863e-12	-1,7042e-4	-6,3442e-4	-0,0141	-0,0138	-0,0128	-3,1438e-7	-1,5338e-3	-2,8549e-3
50	-5,0930e-12	-1,2482e-4	-4,7088e-4	-0,0127	-0,0125	-0,0118	-2,5465e-7	-1,2482e-3	-2,3544e-3
55	-3,8264e-12	-9,4099e-5	-3,5855e-4	-0,0116	-0,0114	-0,0108	-2,1045e-7	-1,0351e-3	-1,9720e-3
60	-2,9473e-12	-7,2670e-5	-2,7902e-4	-0,0106	-0,0105	-0,0100	-1,7684e-7	-8,7204e-4	-1,6741e-3
65	-2,3182e-12	-5,7274e-5	-2,2122e-4	-9,7942e-3	-9,6793e-3	-9,3465e-3	-1,5068e-7	-7,4456e-4	-1,4379e-3
70	-1,8560e-12	-4,5931e-5	-1,7825e-4	-9,0946e-3	-9,0025e-3	-8,7344e-3	-1,2992e-7	-6,4304e-4	-1,2478e-3
75	-1,5090e-12	-3,7393e-5	-1,4568e-4	-8,4883e-3	-8,4133e-3	-8,1943e-3	-1,1318e-7	-5,6089e-4	-1,0926e-3
80	-1,2434e-12	-3,0844e-5	-1,2054e-4	-7,9578e-3	-7,8960e-3	-7,7148e-3	-9,9472e-8	-4,9350e-4	-9,6435e-4
85	-1,0366e-12	-2,5737e-5	-1,0085e-4	-7,4897e-3	-7,4381e-3	-7,2866e-3	-8,8114e-8	-4,3754e-4	-8,5724e-4
90	-8,7328e-13	-2,1698e-5	-8,5211e-5	-7,0736e-3	-7,0301e-3	-6,9021e-3	-7,8595e-8	-3,9056e-4	-7,6690e-4
95	-7,4253e-13	-1,8461e-5	-7,2634e-5	-6,7013e-3	-6,6643e-3	-6,5552e-3	-7,0540e-8	-3,5075e-4	-6,9002e-4
100	-6,3662e-13	-1,5836e-5	-6,2408e-5	-6,3662e-3	-6,3345e-3	-6,2408e-3	-6,3662e-8	-3,1673e-4	-6,2408e-4
105	-5,4994e-13	-1,3686e-5	-5,4010e-5	-6,0631e-3	-6,0357e-3	-5,9545e-3	-5,7744e-8	-2,8741e-4	-5,6710e-4
110	-4,7830e-13	-1,1908e-5	-4,7050e-5	-5,7875e-3	-5,7636e-3	-5,6930e-3	-5,2613e-8	-2,6198e-4	-5,1754e-4
115	-4,1859e-13	-1,0425e-5	-4,1233e-5	-5,5358e-3	-5,5150e-3	-5,4531e-3	-4,8138e-8	-2,3978e-4	-4,7418e-4
120	-3,6842e-13	-9,1785e-6	-3,6335e-5	-5,3052e-3	-5,2868e-3	-5,2323e-3	-4,4210e-8	-2,2028e-4	-4,3602e-4
125	-3,2595e-13	-8,1228e-6	-3,2182e-5	-5,0930e-3	-5,0767e-3	-5,0284e-3	-4,0744e-8	-2,0307e-4	-4,0227e-4
130	-2,8977e-13	-7,2229e-6	-2,8637e-5	-4,8971e-3	-4,8826e-3	-4,8396e-3	-3,7670e-8	-1,8779e-4	-3,7228e-4
135	-2,5875e-13	-6,4511e-6	-2,5593e-5	-4,7157e-3	-4,7028e-3	-4,6644e-3	-3,4931e-8	-1,7418e-4	-3,4551e-4
140	-2,3201e-13	-5,7854e-6	-2,2966e-5	-4,5473e-3	-4,5357e-3	-4,5012e-3	-3,2481e-8	-1,6199e-4	-3,2152e-4
145	-2,0882e-13	-5,2082e-6	-2,0685e-5	-4,3905e-3	-4,3801e-3	-4,3490e-3	-3,0279e-8	-1,5104e-4	-2,9993e-4
150	-1,8863e-13	-4,7053e-6	-1,8696e-5	-4,2441e-3	-4,2347e-3	-4,2067e-3	-2,8294e-8	-1,4116e-4	-2,8044e-4



$\sigma_x(0,y)$ $\sigma_x\left(\frac{a}{2},y\right)$ $\sigma_x(a,y)$	$\sigma_y(0,y)$ $\sigma_y\left(\frac{a}{2},y\right)$ $\sigma_y(a,y)$	$\tau_{xy}(0,y)$ $\tau_{xy}\left(\frac{a}{2},y\right)$ $\tau_{xy}(a,y)$

Fig. 2. Graphical distribution of the stresses in the elastic halfplane in vertical direction

Table 2. The ordinates and the kind of the line influences

x	$\sigma_x(x,0{,}01)$	$\sigma_x\left(x,\frac{a}{2}\right)$	$\sigma_x(x,a)$	$\sigma_y(x,0{,}01)$	$\sigma_y\left(x,\frac{a}{2}\right)$	$\sigma_y(x,a)$	$\tau_{xy}(x,0{,}01)$	$\tau_{xy}\left(x,\frac{a}{2}\right)$	$\tau_{xy}(x,a)$
-60	-1,7684e-6	-8,7204e-4	-1,6741e-3	-4,9122e-14	-6,0558e-6	-4,6503e-5	2,9473e-10	7,2670e-5	2,7902e-4
-55	-2,1045e-6	-1,0351e-3	-1,9720e-3	-6,9571e-14	-8,5544e-6	-6,5190e-5	3,8264e-10	9,4099e-5	3,5854e-4
-50	-2,5465e-6	-1,2482e-3	-2,3544e-3	-1,0186e-13	-1,2482e-5	-9,4175e-5	5,0930e-10	1,2482e-4	4,7087e-4
-45	-3,1438e-6	-1,5338e-3	-2,8549e-3	-1,5525e-13	-1,8936e-5	-1,4098e-4	6,9862e-10	1,7042e-4	6,3442e-4
-40	-3,9789e-6	-1,9287e-3	-3,5245e-3	-2,4868e-13	-3,0136e-5	-2,2028e-4	9,9472e-10	2,4109e-4	8,8113e-4
-35	-5,1969e-6	-2,4955e-3	-4,4421e-3	-4,2424e-13	-5,0930e-5	-3,6262e-4	1,4848e-9	3,5651e-4	1,2692e-3
-30	-7,0736e-6	-3,3482e-3	-5,7296e-3	-7,8595e-13	-9,3005e-5	-6,3662e-4	2,3579e-9	5,5803e-4	1,9099e-3
-25	-1,0186e-5	-4,7087e-3	-7,5698e-3	-1,6297e-12	-1,8835e-4	-1,2112e-3	4,0744e-9	9,4175e-4	3,0279e-3
-20	-1,5915e-5	-7,0491e-3	-0,0102	-3,9789e-12	-4,4057e-4	-2,5465e-3	7,9577e-9	1,7623e-3	5,0930e-3
-15	-2,8294e-5	-0,0115	-0,0136	-1,2575e-11	-1,2732e-3	-6,0272e-3	1,8863e-8	3,8197e-3	9,0408e-3
-10	-6,3662e-5	-0,0204	-0,0159	-6,3662e-11	-5,0930e-3	-0,0159	6,3662e-8	0,0102	0,0159
-5	-2,5465e-4	-0,0318	-0,0102	-1,0186e-9	-0,0318	-0,0407	5,0929e-7	0,0318	0,0204
0	0,0000	0,0000	0,0000	-63,6620	-0,1273	-0,0637	0,0000	0,0000	0,0000
5	-2,5465e-4	-0,0318	-0,0102	-1,0186e-9	-0,0318	-0,0407	-5,0929e-7	-0,0318	-0,0204
10	-6,3662e-5	-0,0204	-0,0159	-6,3662e-11	-5,0930e-3	-0,0159	-6,3662e-8	-0,0102	-0,0159
15	-2,8294e-5	-0,0115	-0,0136	-1,2575e-11	-1,2732e-3	-6,0272e-3	-1,8863e-8	-3,8197e-3	-9,0408e-3
20	-1,5915e-5	-7,0491e-3	-0,0102	-3,9789e-12	-4,4057e-4	-2,5465e-3	-7,9577e-9	-1,7623e-3	-5,0930e-3
25	-1,0186e-5	-4,7087e-3	-7,5698e-3	-1,6297e-12	-1,8835e-4	-1,2112e-3	-4,0744e-9	-9,4175e-4	-3,0279e-3
30	-7,0736e-6	-3,3482e-3	-5,7296e-3	-7,8595e-13	-9,3005e-5	-6,3662e-4	-2,3579e-9	-5,5803e-4	-1,9099e-3
35	-5,1969e-6	-2,4955e-3	-4,4421e-3	-4,2424e-13	-5,0930e-5	-3,6262e-4	-1,4848e-9	-3,5651e-4	-1,2692e-3
40	-3,9789e-6	-1,9287e-3	-3,5245e-3	-2,4868e-13	-3,0136e-5	-2,2028e-4	-9,9472e-10	-2,4109e-4	-8,8113e-4
45	-3,1438e-6	-1,5338e-3	-2,8549e-3	-1,5525e-13	-1,8936e-5	-1,4098e-4	-6,9862e-10	-1,7042e-4	-6,3442e-4
50	-2,5465e-6	-1,2482e-3	-2,3544e-3	-1,0186e-13	-1,2482e-5	-9,4175e-5	-5,0930e-10	-1,2482e-4	-4,7087e-4
55	-2,1045e-6	-1,0351e-3	-1,9720e-3	-6,9571e-14	-8,5544e-6	-6,5190e-5	-3,8264e-10	-9,4099e-5	-3,5854e-4
60	-1,7684e-6	-8,7204e-4	-1,6741e-3	-4,9122e-14	-6,0558e-6	-4,6503e-5	-2,9473e-10	-7,2670e-5	-2,7902e-4

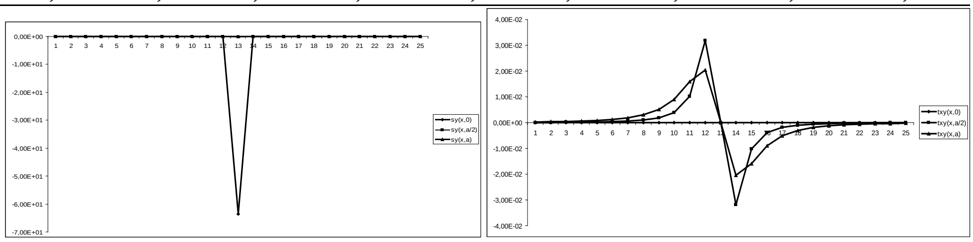


Fig. 3. Graphical distribution of the stresses in the axis elastic halfplane in horizontal direction

From the results obtained it is seen that the normal stresses σ_x in vertical direction have big values in the interval $0 \le y \le 2a$. The normal stresses σ_y in vertical direction have very big values in the interval $0 \le y \le 2a$ and then gradually decrease. The tangential stresses τ_{xy} have big values in the interval $0 \le y \le 2a$, and then gradually decrease. The biggest values of the normal and the tangential stresses in vertical direction are obtained at $y = \dfrac{a}{2}$.

The normal stresses σ_x in horizontal direction have big values in the interval $-6a \le x \le 6a$. The normal stresses σ_y in the horizontal directions have very big values in the interval $-a \le x \le a$, and then gradually decrease. The tangential stresses τ_{xy} have relative big values in the interval $-4a \le x \le 4a$, and then gradually decrease. The biggest values of the normal and the tangential stresses in horizontal direction are obtained at $x = 0$ и $y = 0$.

In vertical direction the normal stresses σ_y are bigger in comparison with the normal stresses σ_x.

The results obtained form stresses line's state in an elastic halfplane.

From the diagrams of stress's distribution in a vertical and horizontal direction, in a selection arbitrary section of elastic halfplane, after integration – analytical of numerical can to make a verification of equilibrium of calculated stresses.

STRESSES LINE INFLUENCES

The expressions of the functions's stresses in a horizontal directions can to interpreter to construct the lines influences of the normal and tangential stresses in elastic halfplane (Fig. 4).

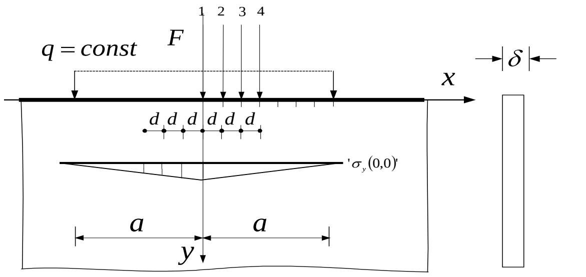


Fig. 4. The expressions of the functions's stresses in a horizontal directions

Here,

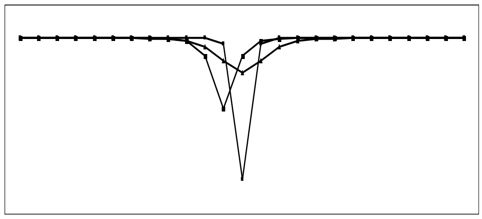
d is the length of the discretization step in horizontal direction; $i = 1,2,3,4...$ – successive situations of the force $F = 1$ in the top edge of the elastic halfplane.

The ordinates and the kind of the line influences, for example $"\sigma_x\left(0{,}01;\frac{a}{2}\right)"$, $"\sigma_y\left(0{,}01;\frac{a}{2}\right)"$, $"\tau_{xy}(0{,}01;\frac{a}{2})"$, etc., are shown in Table 2 and Fig. 3. The ordinates of stresses influences are reports with multiplay of discratization step, suitable to the situation of the forces $F = 1$ and of the stress at the point to from which the line influence refers. For each particular case expressions are written for line influences of the stresses at the point of elastic halfplane presented through stresses line states. For example:

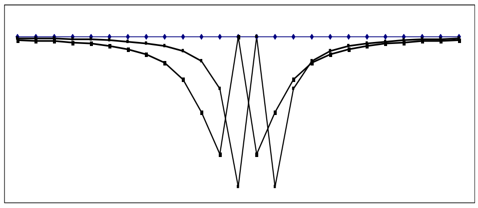
$$"\sigma_x\left(0;\frac{a}{2}\right)" = \sigma_x\left(x;\frac{a}{2}\right), \qquad "\sigma_y\left(\frac{a}{2};\frac{a}{2}\right)" = \sigma_y\left(x - \frac{a}{2};\frac{a}{2}\right),$$

$$"\tau_{xy}\left(-\frac{a}{2};\frac{a}{2}\right)" = \tau_{xy}\left(x + \frac{a}{2};\frac{a}{2}\right).$$

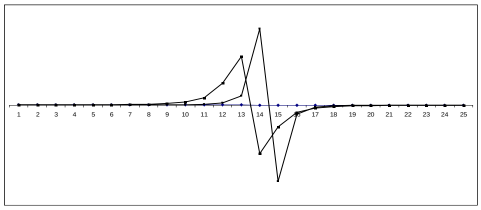
Graphical line influences of the stresses at the selected points of the elastic halfplane are shown in Fig. 5.



$"\sigma_x\left(-\frac{a}{4};\frac{a}{2}\right)"$ $"\sigma_x\left(0;\frac{a}{2}\right)"$ $"\sigma_x\left(\frac{a}{4};\frac{a}{4}\right)"$



$"\sigma_y\left(-\frac{a}{4};\frac{a}{2}\right)"$ $"\sigma_y\left(0;\frac{a}{2}\right)"$ $"\sigma_y\left(\frac{a}{4};\frac{a}{4}\right)"$



$"\tau_y\left(-\frac{a}{4};\frac{a}{2}\right)"$ $"\tau_y\left(0;\frac{a}{2}\right)"$ $"\tau_y\left(\frac{a}{4};\frac{a}{4}\right)"$

Fig. 5. Influences of the stresses at the selected points of the elastic halfplane

From the line influences the stresses at a unspecified point of elastic halfplane at an arbitrary load can be determined. At the load, distributed by arbitrary law, it is known that the arbitrary stress is determines with the expression: $S(x,y) = \int_{q_1}^{q_2} S(x,y)q(x)dx$ where q_1, q_2 are the limits (start, end) of the distributed load, y is a previously determined level in vertical direction at the point at which the stresses are determined through line influences.

At a uniformly distributed load $q = const$ with length *2a*, symmetrically situated about axis

$$y \ \sigma_x\left(0,\frac{a}{2}\right) = \int_{-a}^{a} -\frac{2q}{\pi}x^2 \frac{\frac{a}{2}}{\left[x^2 + \left(\frac{a}{2}\right)^2\right]^2}dx = -\frac{2}{5}\frac{\left(-2 + 5atg(2)\right)}{\pi}q = -0{,}4502q\,;$$

$$\sigma_y\left(0,\frac{a}{2}\right) = \int_{-a}^{a} -\frac{2q}{\pi}\frac{\left(\frac{a}{2}\right)^3}{\left[x^2 + \left(\frac{a}{2}\right)^2\right]^2}dx = -\frac{2}{5}\frac{\left(2 + 5atg(2)\right)}{\pi}q = -0{,}9595q\,;$$

$$\sigma_y\left(\frac{a}{2},\frac{a}{2}\right)=\int_{-a}^{a}-\frac{2q}{\pi}\frac{\left(\frac{a}{2}\right)^3}{\left[x^2+\left(\frac{a}{2}\right)^2\right]^2}dx=-\frac{2}{5\pi}q=-0{,}9022q\,;$$

$$\tau_{xy}\left(0,\frac{a}{2}\right)=\int_{-a}^{a}-\frac{2q}{\pi}x\frac{\left(\frac{a}{2}\right)^2}{\left[x^2+\left(\frac{a}{2}\right)^2\right]^2}dx=0\,;\ \tau_{xy}\left(\frac{a}{2},\frac{a}{2}\right)=\int_{-a}^{a}-\frac{2q}{\pi}x\frac{\left(x+\frac{a}{2}\right)^2}{\left[(x+\frac{a}{2})^2+\left(\frac{a}{2}\right)^2\right]^2}dx=-\frac{2}{5\pi}q=-0{,}1273q\,.$$

At a load with length *a*, distributed by triangle law with zero of a triangle at the $x=0$ (Fig. 6), the stresses, determined through line influences, are of the type:

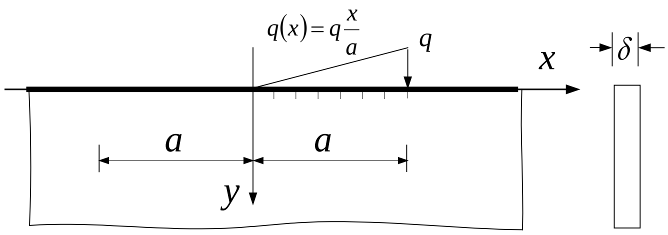


Fig. 6. Type of the stresses, determined through line influences

$$\sigma_x\left(0,\frac{a}{2}\right)=\int_{0}^{a}-\frac{2}{\pi}x^2\frac{\frac{a}{2}}{\left[x^2+\left(\frac{a}{2}\right)^2\right]^2}\left(\frac{qx}{a}\right)dx=\left(\frac{-(1+15\ln 5)}{10\pi}+\frac{(1+2\ln 5)}{2\pi}\right)q=-0{,}1288q\,;$$

$$\sigma_y\left(0,\frac{a}{2}\right)=\int_{0}^{a}-\frac{2}{\pi}\frac{\left(\frac{a}{2}\right)^3}{\left[x^2+\left(\frac{a}{2}\right)^2\right]^2}\left(\frac{qx}{a}\right)dx=-\frac{2}{5\pi}q=-0{,}1273q\,;$$

$$\sigma_y\left(\frac{a}{2},\frac{a}{2}\right)=\int_{0}^{a}-\frac{2}{\pi}\frac{\left(\frac{a}{2}\right)^3}{\left[\left(x-\frac{a}{2}\right)^2+\left(\frac{a}{2}\right)^2\right]^2}\left(\frac{qx}{a}\right)dx=-\frac{1}{8}\left(\frac{(-2+\pi)}{\pi}+1\right)q=-0{,}0908q\,;$$

$$\tau_{xy}\left(0,\frac{a}{2}\right)=\int_{0}^{a}-\frac{2}{\pi}x\frac{\left(\frac{a}{2}\right)^2}{\left[x^2+\left(\frac{a}{2}\right)^2\right]^2}\left(\frac{qx}{a}\right)dx=\frac{2-5arctg2}{10\pi}q=-0{,}1126q\,;$$

$$\tau_{xy}\left(\frac{a}{2},\frac{a}{2}\right)=\int_{0}^{a}-\frac{2}{\pi}\left(x-\frac{a}{2}\right)\frac{\left(\frac{a}{2}\right)^2}{\left[(x-\frac{a}{2})^2+\left(\frac{a}{2}\right)^2\right]^2}\left(\frac{qx}{a}\right)dx=\left(\frac{1}{\pi}-\frac{1}{2\pi}arctg2\right)q=0{,}1421q\,.$$

The results obtained through line influences coincide exactly with the values of the stresses in the elastic isotropic halfspace, obtained from the solution under the effect of indicated distributed loads.

CONCLUSIONS

With more concentrated forces the diagrams of stresses are superimposed. In a similar way the stresses at different points of an elastic isotropic semi-space with other acting loads can be determined.

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