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Numerical study of the non-linear radiation heat transfer problem for the flow of a second-grade fluid

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Radiation effects in the two-dimensional flow of an electrically conducting second-grade fluid are examined. Non-linear radiative heat flux is considered in the formulation of the energy equation. Viscous dissipation effects are retained. The developed nonlinear differential systems are solved numerically using the shooting method with a fourth-fifth order Runge-Kutta integration procedure. The solutions are validated with the built-in numerical solver bvp4c of the software MATLAB. The dimensionless expressions of skin friction coefficient and rate of heat transfer at the sheet are evaluated and discussed.

Keywords: Thermal radiation; Second-grade fluid; Heat transfer; Shooting method; Non-linear Rosseland approximation

INTRODUCTION

The study of viscoelastic boundary layer flows due to the movement of inextensible surfaces is important in many manufacturing processes. A number of technical processes concerning polymers involve the cooling of continuous strips or filaments by drawing them through a quiescent fluid. The thin polymer sheet constitutes a continuously moving surface with a non-uniform velocity through an ambient fluid. In these cases the properties of the final product depend to a great extent on the rate of cooling, which is governed by the structure of the boundary layer near the moving strip. Due to the entrainment of the ambient fluid, this boundary layer is different from that in the Blasius [1] flow past a fixed flat plate. Crane [2] was probably the first to discover the flow due to a stretching surface in an otherwise ambient fluid. Since then many authors have studied various aspects of this problem such as the effects of surfaces mass transfer, magnetic field, arbitrary stretching velocity, variable wall temperature or heat flux (Gupta and Gupta [3], Chakrabarti and Gupta [4], Grubka and Bobba [5], Banks [6], Chen and Char [7], Ali [8], Pop and Na [9], Magyari and Keller [10], Liao and Pop [11], Kumari and Nath [12], Hayat *et al*. [13-15], Mustafa *et al*. [16,17], Zheng *et al.* [18-20], Liu *et al.* [21], etc. The thermal radiation effect in such flow configurations is prominent in nuclear power plants, satellites and space vehicles, in combustion appliances such as fires, furnaces, IC engines, ship compressors, solar radiation buildings, etc. Influence of thermal radiation on the steady incompressible flow of a viscoelastic fluid with constant suction has been discussed by Raptis and Perdikis [22]. Sedeek [23] and Raptis *et al*. [24] examined the thermal radiation effect on the boundary layer flow of an electrically conducting viscous fluid. Bataller [25] examined the radiation effects in the Blasius flow of a viscous fluid. Elbeshbeshy and Emam [26] discussed the thermal radiation effects on the unsteady flow due to a stretching sheet immersed in a porous medium. Homotopic solutions for a unsteady mixed convection flow of a Jeffrey fluid with thermal radiation have been provided by Hayat and Mustafa [27]. Motsumi and Makinde [28] investigated the radiation effects on the incompressible flow of a nanofluid with viscous dissipation. Flow and heat transfer of a MHD viscous fluid over an unsteady stretching surface with radiation heat flux are examined by Zheng *et al.* [29]. In another paper, Zheng *et al.* [30] discussed the buoyancy lift effects on the mixed flow and radiation heat transfer of a micropolar fluid towards a vertical permeable plate.

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Recently the flow analysis of non-Newtonian fluids has received remarkable attention due to its relevance in various processes such as plastic manufacture, performance of lubricants, application of paints, polymer processing, food processing and movement of biological fluids. In particular, the boundary layer flow of a second-grade fluid is widely discussed. Numerical investigation of the mass transfer effects in the flow of an electrically conducting second-grade fluid has been performed by Cortell [31]. Mixed convection flow of a second-grade fluid past a vertical flat surface with variable surface temperature has been investigated by Mushtaq *et al*. [32]. Hayat *et al*. [33] examined the effects of thermal radiation and viscous dissipation in the Blasius flow of a second-grade fluid. Three-dimensional boundary layer flow analysis of a second- grade fluid has been addressed by Nazar and Latip [34]. Exact solutions for the hydro-magnetic flow and heat transfer in a second-grade fluid with heat generation/absorption have been obtained by Abel *et al*. [35]. Nazar *et al*. [36] discussed the Stokes second problem for a second-grade fluid. Jamil *et al*. [37] examined the flow of a second-grade fluid due to constantly accelerated shear stresses. Homotopic solutions for a squeezing flow of a second-grade fluid between parallel disks have been computed by Hayat *et al.* [38]. Perturbation analysis for a flow of modified second-grade fluid over a porous plate has been performed by Pakdermili *et al.* [39]. Steady laminar boundary layer flow of a second-grade fluid in the presence of thermophoresis effects has been examined by Olajuwon [40].

In some recent papers the heat transfer characteristics have been investigated using non-linear Rosseland approximation for thermal radiation (see Pantokratoras and Fang [41], Mushtaq *et al.* [42], Cortell [43] and Mushtaq *et al.* [44]). The present work deals with the influence of non-linear thermal radiation on the flow of an electrically conducting second-grade fluid due to a stretching sheet. The developed mathematical problems were solved for the numerical solution through a shooting method. It is important to point out that computation of either analytic or numerical solutions of the classical Navier–Stokes equations (characterizing complex flow mechanics) [45-48] is often handy for the researchers. Graphs showing the behavior of various parameters are sketched and analyzed.

PROBLEM FORMULATION

We consider the steady flow of an incompressible second-grade fluid over a stretching sheet situated at $y = 0$. Let $U_w = ax$ be the velocity of the stretching sheet where $a > 0$ is constant. A uniform transverse magnetic field of strength B_0 is applied normal to the flow. The induced magnetic field is neglected under the assumption of small magnetic Reynolds' number. The boundary layer equations governing the steady two-dimensional stagnation-point flow of second- grade fluid are [31-33]:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{1}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial v}{\partial y}=\nu\frac{\partial^2 u}{\partial y^2}-\frac{\sigma B_0^2}{\rho}u+\frac{\alpha_1}{\rho}\left(\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial y^2}+u\frac{\partial^3 u}{\partial x\partial y^2}+v\frac{\partial^3 u}{\partial y^3}+\frac{\partial u}{\partial y}\frac{\partial^2 v}{\partial y^2}\right), \tag{2}$$

where ν is the kinematic viscosity, σ is the electrical conductivity of the fluid, B_0 is uniform magnetic field along the y-axis, $\alpha_1(\geq 0)$ is the material fluid parameter of the second-grade fluid, u and v are the velocity components in x- and y-directions, respectively. The boundary conditions in the present problem are

$$u=U_w(x)=ax,\qquad v=0\quad \text{at}\ y=0,\qquad u\to 0\quad \text{as}\ y\to\infty. \tag{3}$$

Introducing the following variables

$$\eta=\sqrt{\frac{a}{\nu}}y,\quad u=axf'(\eta),\quad v=-\sqrt{\nu a}f(\eta). \tag{4}$$

Eq. (1) is identically satisfied and Eqs. (2) and (3) become

$$f'''+ff''-f'^2-Mf'+\beta(2f'f'''-ff^{iv}-f''^2)=0 \tag{5}$$

$$f(0)=0,\ f'(0)=1,\ f'(+\infty)\to 0, \tag{6}$$

where $M=\sigma B_0^2/\rho a$ is the magnetic parameter and $\beta=\alpha_1 a/\rho\nu$ is viscoelastic parameter. It interesting to note that Eq. (5) subjected to the boundary conditions (6) admits a closed form exact solution of the following form (see Cortell [31]).

$$f(\eta)=\frac{1-e^{-b\eta}}{b};\quad b=\sqrt{\frac{1+M}{1+\beta}}. \tag{7}$$

Heat transfer analysis

Under usual boundary layer assumptions, the energy equation in the presence of thermal radiation and viscous dissipation effects is given by

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\alpha\frac{\partial^2 T}{\partial y^2}+\frac{\nu}{C_p}\left(\frac{\partial u}{\partial y}\right)^2-\frac{1}{\rho C_p}\left(\frac{\partial q_r}{\partial y}\right)+\frac{\alpha_1}{\rho C_p}\left(u\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial x\partial y}+v\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y^2}\right), \tag{8}$$

where T is the temperature, α is the thermal diffusivity, C_p is the specific heat at constant pressure and q_r is the radiative heat flux. Using the

Rosseland approximation for thermal radiation and applying to optically thick media, the radiative heat flux in Eq. (8) is given by [49]

$$q_r = -\frac{4\sigma^*}{3k^*}\frac{\partial T^4}{\partial y} = -\frac{16\sigma^*}{3k^*}T^3\frac{\partial T}{\partial y}\ , \tag{9}$$

where σ^* and $\kappa^*$are the Stefan-Boltzman constant and the mean absorption coefficient, respectively. Now (8) can be expressed as

$$\begin{aligned} u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} &= \frac{\partial}{\partial y}\left[\left(\alpha + \frac{16\sigma^* T^3}{3\rho C_p k^*}\right)\frac{\partial T}{\partial y}\right] \\ &+ \frac{\nu}{C_p}\left(\frac{\partial u}{\partial y}\right)^2 + \frac{\alpha_1}{\rho C_p}\left(u\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial x \partial y} + v\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y^2}\right). \end{aligned} \tag{10}$$

It is worth mentioning here that in the previous studies on thermal radiation (see [22-30] and various references therein), T^4 in Eq. (9) was expanded about the ambient temperature T_∞. However in the subsequent subsection this was avoided.

Constant wall temperature (CWT)

The relevant boundary conditions in this situation are

$$T = T_w \quad \text{at } y = 0; \quad T \to T_\infty \text{ as } y \to \infty, \tag{11}$$

with $T_w > T_\infty$ and T_w and T_∞ are the sheet's temperature and the ambient fluid's temperature, respectively. Defining the non-dimensional temperature $\theta(\eta) = (T - T_\infty)/(T_w - T_\infty)$ and also $T = T_\infty(1 + (\theta_w - 1)\theta)$ with $\theta_w = T_w/T_\infty$ (temperature ratio parameter), the first term on the right hand side of Eq. (10) can be written as $\alpha\partial/\partial y[\partial T/\partial y(1 + R_d(1 + (\theta_w - 1)\theta)^3)]$, where $R_d = 16\sigma^* T_\infty^3/3kk^*$ denotes the radiation parameter for the CWT case, and $R_d = 0$ provides no thermal radiation effect. The latter expression can be further reduced to:

$$\frac{a(T_w - T_\infty)}{Pr}[(1 + R_d(1 + (\theta_w - 1)\theta)^3)\theta']', \tag{12}$$

where $Pr = \nu/\alpha$ is the Prandtl number. Eq. (14) takes the following form

$$\begin{aligned} &[(1 + R_d(1 + (\theta_w - 1)\theta)^3)\theta']' \\ &= -Pr\left(f\theta' + E_c^*\left[f''^2 + \beta\left(f'f''^2 - ff''f'''\right)\right]\right), \end{aligned} \tag{13}$$

with boundary conditions

$$\theta(0) = 1, \qquad \theta(+\infty) \to 0, \tag{14}$$

where $E_c^* = U_w^2/C_p(T_w - T_\infty)$ is the local Eckert number. We notice that x- coordinate could not be eliminated from the energy equation. Thus we look for the availability of local similarity solutions. Using Eq. (7), (13) becomes

$$\begin{aligned} &[(1 + R_d(1 + (\theta_w - 1)\theta)^3)\theta']' \\ &= -Pr\left[\left(\frac{1 - e^{-b\eta}}{b}\right)\theta' + E_c^* b^2(1 + \beta)e^{-2b\eta}\right]. \end{aligned} \tag{15}$$

The heat transfer rate at the sheet is defined as

$$\begin{aligned} q_w &= -k\left(\frac{\partial T}{\partial y}\right)_{y=0} + (q_r)_w \\ &= -k(T_w - T_\infty)\sqrt{\frac{a}{\nu}}[1 + R_d\theta_w^3]\theta'(0), \end{aligned} \tag{16}$$

and with the help of the local Nusselt number $Nu_x = xq_w/k(T_w - T_\infty)$, one obtains

$$\frac{Nu_x}{\sqrt{Re_x}} = -[1 + R_d\theta_w^3]\theta'(0). \tag{17}$$

Prescribed surface temperature (PST)

The boundary conditions in this case are

$$\begin{aligned} T = T_w = T_\infty + cx^2 \quad &\text{at } y = 0; \\ T \to T_\infty \quad &\text{as } y \to \infty, \end{aligned} \tag{18}$$

where $c > 0$ is a constant and Eq. (10) reduces to

$$\begin{aligned} &(1 + R_d)\theta'' \\ &= -Pr\left[\begin{matrix} f\theta' - 2f'\theta \\ +E_c\left[f''^2 + \beta\left(f'f''^2 - ff''f'''\right)\right] \end{matrix}\right], \end{aligned} \tag{19}$$

with the boundary conditions (18). Here $E_c = a^2/cC_p$ is the constant Eckert number. Using Eq. (7), (19) becomes

$$\begin{aligned} &(1 + R_d)\theta'' \\ &= -Pr\left[\begin{matrix} \left(\frac{1 - e^{-b\eta}}{b}\right)\theta' - 2\theta e^{-b\eta} \\ +E_c b^2(1 + \beta)e^{-2b\eta} \end{matrix}\right]. \end{aligned} \tag{20}$$

Here the heat transfer rate at the sheet becomes

$$\begin{aligned} q_w &= -k\left(\frac{\partial T}{\partial y}\right)_{y=0} + (q_r)_w \\ &= -kcx^2\sqrt{\frac{a}{\nu}}[1 + R_d]\theta'(0), \end{aligned} \tag{21}$$

and using the definition of reduced Nusselt number one obtains

$$\frac{Nu_x}{\sqrt{Re_x}} = -[1 + R_d]\theta'(0). \tag{22}$$

Numerical Method

We have solved Eqs. (15) and (20) with the boundary conditions (14) by the shooting method using fourth-fifth order Runge-Kutta integration technique. The governing equations are reduced to a system of first order equations with boundary conditions (14) as

CWT case

$$\begin{cases} \dfrac{d\theta}{d\eta} = P; \\ \dfrac{dP}{d\eta} = \dfrac{\left[\begin{array}{c} -\Pr\left(\left(\frac{1-e^{-b\eta}}{b}\right)P + E_c^* b^2(1+\beta)e^{-2b\eta}\right) \\ -3R_d P^2(\theta_w - 1)(1+(\theta_w-1)\theta)^2 \end{array}\right]}{1+R_d(1+(\theta_w-1)\theta)^3} \end{cases} \tag{23}$$

PST case

$$\begin{cases} \dfrac{d\theta}{d\eta} = P; \\ \dfrac{dP}{d\eta} = \dfrac{\left[Pr\left(\begin{array}{c} 2\theta e^{-b\eta} - \left(\frac{1-e^{-b\eta}}{b}\right)P \\ -E_c b^2(1+\beta)e^{-2b\eta} \end{array}\right)\right]}{1+R_d}, \end{cases} \tag{24}$$

with boundary conditions

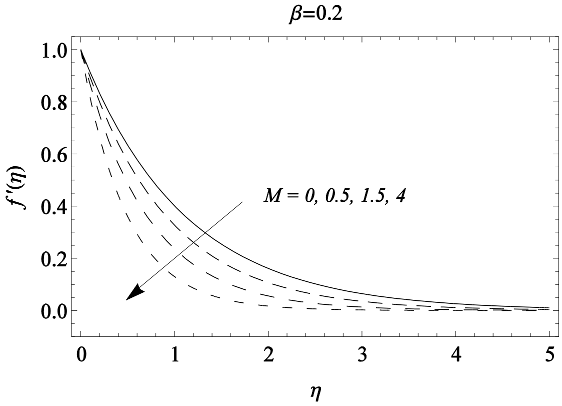
$$\theta(0) = 1, \qquad \theta(\infty) \to 0. \tag{25}$$

In order to integrate Eqs. (23) and (24) we require a value for $P(0)$ i.e $\theta'(0)$, however no such value is given at the boundary. Due to this reason, suitable values of $P(0)$ are guessed and then integration is carried out. The values of $P(0)$ are iteratively obtained such that solutions satisfy the boundary condition at large η with an error less than 10^{-7}.

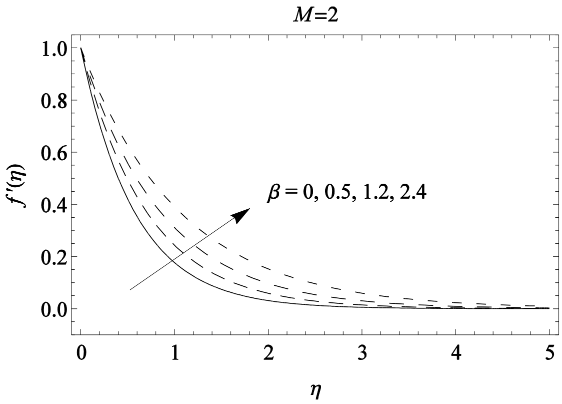
NUMERICAL RESULTS AND DISCUSSION

This section discusses the behavior of embedded physical parameters on the dimensionless velocity and temperature functions. Influence of magnetic parameter M on the velocity is depicted in Fig. 1(a). It is noticed that increasing magnetic field strength restricts the flow and consequently thins the momentum boundary layer. Fig. 1(b) indicates an increase in the velocity and the boundary layer thickness when the viscoelastic effects strengthen.

Fig. 2 is plotted to perceive the effects of radiation parameter on the temperature θ. The results are given for both CWT and PST cases. In contrast to the linear radiation heat transfer problem, even a minor variation in the radiation parameter greatly influences the temperature and the thermal boundary layer thickness. The increase in temperature distribution with the radiation parameter is significant in the CWT case when compared with the PST case.

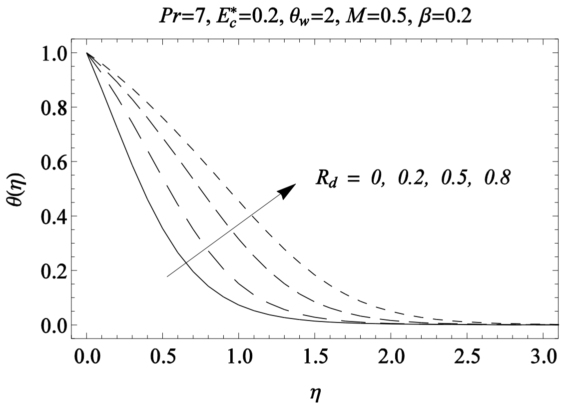


(a) CWT

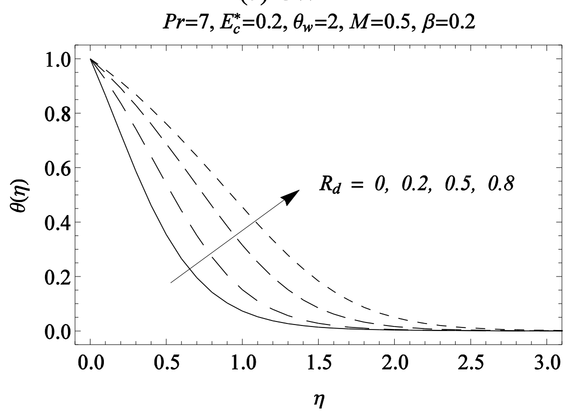


(b) PST

Fig.1. Velocity field f' for different values of magnetic parameter M and viscoelastic parameter β.



(a) CWT



(b) PST

Fig. 2. Temperature profiles for different values of the radiation parameter R_d.

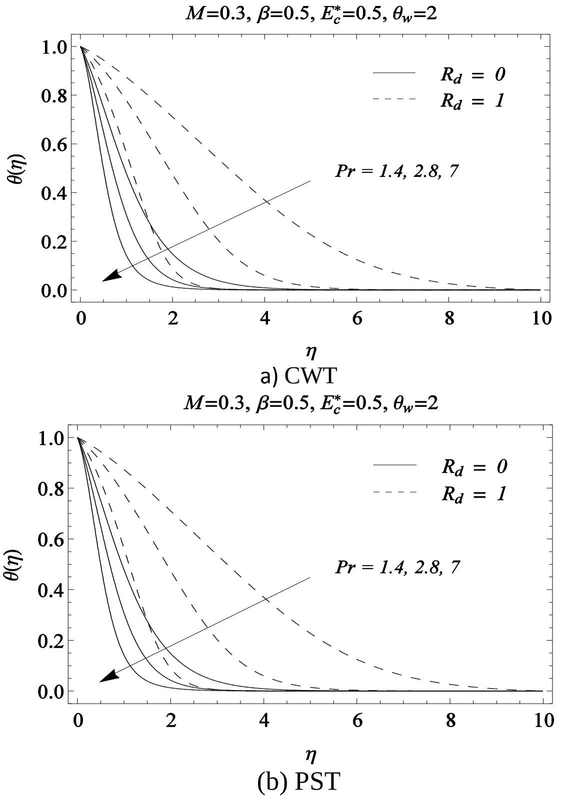


Fig. 3. Temperature profiles for different values of Pr

The effect of Prandtl number Pr on the temperature is shown in Fig. 3.

Increase in Pr may be regarded as a decrease in the thermal diffusivity and consequently a thinner thermal boundary layer is expected for a greater Pr. Specifically $Pr = 0.72, 1, 7$ corresponds to air, electrolyte solution such as mixture of salt and water and water, respectively. We observed that the profiles get closer to the boundary as Pr increases indicating a diminution in the thermal boundary layer thickness. We also noticed that temperature profiles show large deviation with the variation of Pr for a sufficiently strong thermal radiation effect ($R_d = 1$).

The influence of Eckert number Ec on the temperature for both CWT and PST cases is observed in Fig. 4. Here the profiles are computed with and without thermal radiation effects. When $R_d = 0$ the temperature θ first increases to a maximum value and then smoothly descends to zero value as $\eta \to \infty$. Moreover, the temperature θ is an increasing function of E_c.

Fig. 5 illustrates the behavior of second-grade fluid parameter β on the thermal boundary layer. In accordance with Abel *et al.* [39], the temperature θ is a decreasing function of β. In other words, the fluid becomes colder as normal stress differences are increased. Numerical values of the dimensionless heat transfer rate at the sheet for

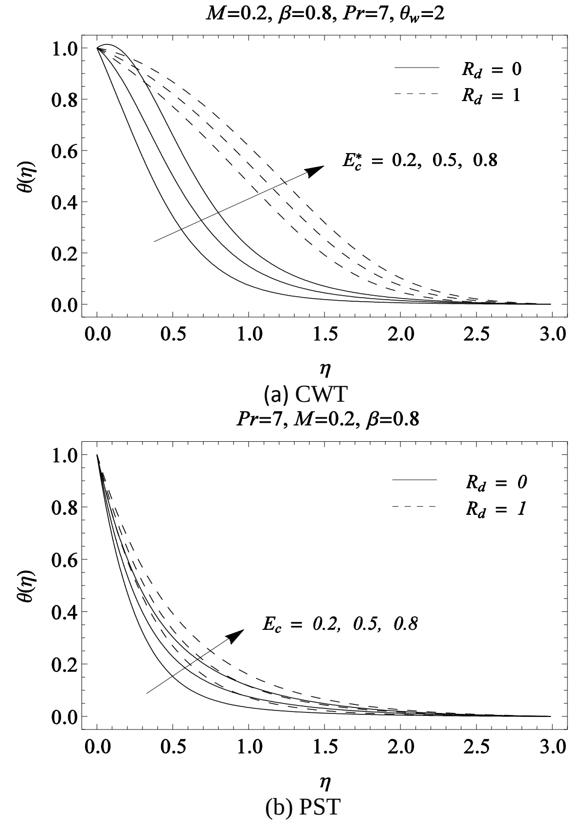


Fig. 4: Temperature profiles for different values of the Eckert number.

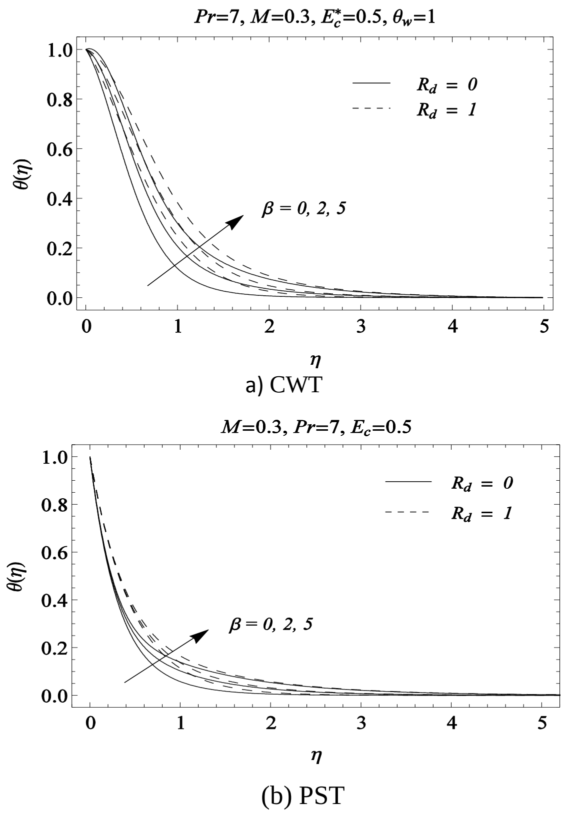


Fig. 5. Temperature profiles for different values of the viscoelastic parameter β.

various parametric values are provided in tables 1 and 2.

The results are presented with and without thermal radiation effects. The results are also compared with those obtained through the built-in numerical solver *bvp4c* of the software MATLAB and found in an excellent agreement. The Nusselt number $(-\theta'(0))$ is positive for all values of the parameters. This is because the fluid is colder than the solid wall and heat, therefore, flows from the stretching sheet to the fluid. Regardless of the values of other parameters the magnitude of the local Nusselt number is larger in the absence of thermal radiation effects ($R_d = 0$). The magnitude of the local Nusselt number is an increasing function of the second-grade fluid parameter. However, it slightly decreases as the magnetic field effects intensify. We have earlier observed in the graphical results that profiles become increasingly steeper when Pr is increased. The Nusselt number, being proportional to the initial slope, increases with an increase in Pr. That's why the wall heat transfer rates are larger in the case of liquids when compared with gases such as air, hydrogen, etc. Moreover, the behavior of the Eckert number E_c on the dimensionless heat transfer rate is similar to that of Pr in a qualitative sense.

CONCLUSIONS

Thermal radiation effects on the flow of an electrically conducting second-grade fluid are investigated. Heat transfer analysis data were considered for two different heating processes, namely, (i) a sheet with a constant wall temperature (CWT) and (ii) a sheet with the prescribed surface temperature (PST). The temperature function in the

Table 1. Heat transfer rate at the wall $\theta'(0)$ in the CWT case for various parametric values

Pr	θ_w	E_c^*	M	β	R_d	$f''(0)$	$\theta'(0)$	
							bvp4c	Shooting
7	2	0.2	0	0	1	-1	-0.37106	-0.37106
					0		-1.45808	-1.45806
				0.5	1	-0.81649	-0.38379	-0.38380
					0		-1.44118	-1.44116
			0.5	1	1	-0.86602	-0.34539	-0.34539
					0		-1.20864	-1.20863
				1.5	1	-0.77459	-0.34809	-0.34809
					0		-1.18001	-1.17999
			1.5	2	1	-0.91287	-0.27540	-0.27540
					0		-0.75827	-0.75826
		0.5		3	1	-0.79056	0.01436	0.01436
					0		1.20696	1.20698
0.72	1	0.2	0.5	0.5	1	-1	-0.22543	-0.22544
					0		-0.36975	-0.36975
			1.5		1	-1.2909	-0.16740	-0.16740
					0		-0.28331	-0.28332

Table 2. Heat transfer rate at the wall $\theta'(0)$ in the PST case for various parametric values

Pr	E_c	M	β	R_d	$f''(0)$	$\theta'(0)$	
						bvp4c	Shooting
7	0.2	0	0	1	-1	-2.56119	-2.56120
				0		-3.69545	-3.69550
			0.5	1	-0.81649	-2.59471	-2.59472
				0		-3.72385	-3.72381
		0.5	1	1	-0.86602	-2.49538	-2.49542
				0		-3.57175	-3.57177
			1.5	1	-0.77459	-2.50739	-2.50744
				0		-3.58013	-3.58019
	0	1.5	2	1	-0.91287	-2.75488	-2.75489
				0		-3.99382	-3.99388
			3	1	-0.79056	-2.78670	-2.78673
				0		-4.02502	-4.02509
0.72	0.5	0.25	1	1	-0.79056	-0.64036	-0.640370
				0		-0.96354	-0.96354
		0.5	2	1	-0.70710	-0.63373	-0.63374
				0		-0.93594	-0.93596

radiation term of the energy equation is not further expanded by Taylors' series about the ambient temperature in the CWT case. The key points of the present study can be summarized as follows:

1. The presence of magnetic field creates a bulk known as Lorentz force which opposes the fluid velocity and, as a consequence, boundary layer thins as the strength of magnetic field increases. Moreover, the temperature θ is an increasing function of the magnetic parameter M. On the other hand, the magnitude of velocity and the boundary layer thickness are increasing functions of the second-grade fluid parameter β.
2. Temperature θ, being a strong function of the radiation parameter in the CWT case, appreciably increases in the CWT case when compared with the PST case.
3. A significant reduction in the temperature function is observed when the Prandtl number Pr is increased for sufficiently stronger thermal radiation effect. This outcome is similar in both CWT and PST cases. Moreover, the rate of heat transfer at the sheet enhances when Pr is increased.
4. The magnitude of Nusselt number $\theta'(0)$ increases with an increase in the viscoelastic effects and magnetic field strength.
5. The present work for the case of Newtonian fluid can be recovered by setting $\beta = 0$.

REFERENCES

1. H. Blasius, *Z. Math. Phys.*, **56,** 1 (1908).
2. L. J. Crane, *J. Appl. Math. Phys. (ZAMP)*, **21,** 645 (1970).
3. P. S. Gupta, A. S. Gupta, *J. Chem. Eng*., **55,** 744 (1977).
4. A. Chakrabarti, A. S. Gupta, *Quart. Appl. Math*., **37,** 73 (1979).
5. L. J. Grupka, K. M. Bobba, *ASME J. Heat Transfer*, **107,** 248 (1985).
6. W. H. H. Banks, *J. Méc. Theor. Appl.*, **2,** 375 (1983.
7. C. K. Chen, M. I. Char, *J. Math. Anal. Appl.*, **135,** 568 (1988).
8. M. E. Ali, *Int. J. Heat Mass Transfer*, **16,** 280 (1995).
9. I. Pop, T. Y. Na, *Mech. Res. Comm*., **25** 263 (1998).
10. E. Magyari, B. Keller, *Eur. J. Mech. B-Fluids*, **19**, 109 (2000).
11. S. J. Liao, I. Pop, *Int. J. Heat Mass Transf.*, **47** 75 (2004).
12. M. Kumari, G. Nath, *Commun. Nonlinear Sci. Numer. Simulat.*, **14**, 3339 (2009).
13. T. Hayat, M. Mustafa, M. Sajid, *Z. Naturforsch.*, **64** 827 (2009).
14. T. Hayat, M. Mustafa, S. Asghar, *Nonlinear Anal.: Real World Applications*, **11,** 3186 (2010).
15. T. Hayat, M. Mustafa, A. A. Hendi, *Appl. Math. Mech*., **32,** 167 (2011).
16. M. Mustafa, T. Hayat and A. A. Hendi, *ASMEJ. Appl. Mech.*, **79**, 1 (2012).
17. M. Mustafa, T. Hayat, S. Obaidat, *Z. Naturforsch*., **67a,** 70 (2012).
18. L. Zheng, J. Niu, X. Zhang and Y. Gao, *Math. Comp. Mod.*, **56**, 133 (2012).
19. L. Zheng, X. Jin, X. Zhang, J. Zhang, *Acta Mechan. Sini.*, **29,** 667 (2013).
20. L. Zheng, N. Liu, X. Zhang, *J. Heat Transf.*, **135,** 031705 (2013).
21. I. C. Liu, H. H. Wang and Y. F. Peng, *Chem. Eng. Comm.*, **200** 253 (2013).
22. A. Raptis and C. Perdikis, *ZAMP*, **78,** 277 (1998).
23. M. A. Seddeek, *Int. J. Heat Mass Transfer*, **45**, 931 (2002).
24. A. Raptis, C. Perdikis, H. S. Takhar, *Appl. Math. Comput.*, **153**, 645 (2004).
25. R. C. Bataller, *Appl. Math. Comput*., 198, 333 (2008).
26. E. M. A. Elbeshbeshy, T. G. Emam, *Therm. Sci.*, **15**, 477 (2011).
27. T. Hayat, M. Mustafa, *Zeitschrift für Naturforschung*, **65a**, 711 (2010.
28. T. G. Motsumi, O. D. Makinde, *Phys. Scr*., **86**, doi:10.1088/0031 8949/86/04/045003 (2012).
29. L. Zheng, L. Wang, X. Zhang, L. Ma, *Chem. Eng. Comm.*, **199**, 1 (2012).
30. L. Zheng, N. Liu, J. Niu, X. Zhang, *J. Porous Media*, **16,** 575 (2013).
31. R. Cortell, *Chem. Eng. Process*., **46**, 721 (2007.
32. M. Mushtaq, S. Asghar, M. A. Hossain, *Heat Mass Transf.*, **43**, 1049 (2007).
33. T. Hayat, M. Mustafa, M. Sajid, *Z. Naturforsch*, **64a,** 827 (2009).
34. R. Nazar, N. A. Latip, *Eur. J. Sci. Res.*, **29**, 509 (2009).
35. M. S. Abel, N. Mahesha and S. B. Malipatil, *Chem. Eng. Comm.*, **198**, 191 (2010).
36. M. Nazar, C. Fetecau, D. Vieru, C. Fetecau, *Nonlinear Anal.: Real World Applications*, **11,** 584 (2010.
37. M. Jamil, A. Rauf, C. Fetecau and N. A. Khan, *Commun. Nonlinear Sci. Numer. Simulat.*, **16,** 1959-(2011).
38. T. Hayat, A. Yousaf, M. Mustafa, S. Obaidat, *Int. J. Num. Meth. Fluids*, **69**, 399 (2012),.
39. M. Pakdermili, T. Hayat, M. Yurusoy, S. Abbasbandy, S. Asghar, *Nonlinear Anal. RWA*, **12,** 1774 (2011).
40. B. I. Olajuwon, *Int. Comm. Heat Mass Transf*., **38**, 377 (2011).
41. A. Pantokratoras, T. Fang, *Physic. Scrip*., **87**, 015703 (2013).
42. A. Mushtaq, M. Mustafa, T. Hayat, A. Alsaedi, *J. Aerosp. Engg.*, **27**, DOI: 10.1061/(ASCE)AS.1943-5525.0000361, (2014).
43. R. Cortell, *J. King Saud University-Science*, **26** (2014) doi.org/10.1016/j.jksus.2013.08.004.
44. A. Mushtaq, M. Mustafa, T. Hayat, A. Alsaedi, *J. Taiwan Inst. Chem. Eng.* ,**45,** 1176 (2014).

45. M. M. Rashidi, S. A. Mohimanian Pour, T. Hayat, S. Obaidat, *Comp. Fluids*, **54**, 1 (2012).
46. M. M. Rashidi, M. T. Rastegari, M. Asadia, O. Anwar Bég, *Chem. Eng. Comm.*, **199**, 231 (2012).
47. O. A. Beg, M. M. Rashidi, T. A. Beg, M. Asadi, *J. Mech. Med. Bio.*, **12**, 1250051 (2012).
48. M. M. Rashidi, S. Abelman, N. FreidooniMehr, *Int. J. Heat Mass Transfer*, **62**, 515 (2013).
49. S. Rosseland, Astrophysik und Atomtheoretische Grundlagen, Springer, Berlin, 1931.

ЧИСЛЕНО ИЗСЛЕДВАНЕ НА НЕЛИНЕЙНИ ПРОБЛЕМИ НА РАДИАЦИОННОТО ТОПЛОПРЕНАСЯНЕ ПРИ ТЕЧЕНИИЕТО НА ФЛУИДИ ОТ ВТОРА СТЕПЕН

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(Резюме)

Изучени са ефектите на радиационното топлопренасяне в двумерно течение в електропроводяща течност от втора степен. Нелинейният радиационен топлинен поток се отчита при формулирането на уравнението за енегията. Ефектите на вискозна дисипация на енергията са взети под внимание. Системата от нелинейни диференциални уравнения е решена числено по метода на прострелването с интеграционна процедура по Рунге-Кута от четвърта степен. Решенията са проверени с вградена числена процедура bvp4c от софтуера MATLAB. Пресметнати са коефициента на триене и скоростта на топлопренасяне.