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On matrices of coefficients of electromagnetic and elastic waves propagating in anisotropic media

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Thermoelasticity describes a wide range of phenomena and generalizes the classical theory of elasticity and the theory of heat conductivity. Thermoelastic and electromagnetic waves propagation in anisotropic media is of the most interest at present. Within the bounds of this area, based on the use of physical-mechanical properties of anisotropic media bound heat and mechanical fields are being studied.

The article is devoted to the study of thermoelastic wave propagation in anisotropic media of hexagonal systems in the case of the second order axis symmetry and heterogeneity along X-axis. In the article, by means of analytical matricant method, a set of motion equations of thermoelastic media is reduced to an equivalent set of first-order differential equations.

The structures of the matrices of the coefficients of the constitutive equations and the structure of the matrix for waves of an acoustic and electromagnetic coupled field in thermoelastic, piezoelectric, piezomagnetic and magnetoelectric anisotropic media are presented.

Keywords: Anisotropic medium, thermoelastic and electromagnetic waves, matricant.

INTRODUCTION

The dynamical theory of thermoelasticity is the study of dynamical interaction between thermal and mechanical fields in solid bodies and is of high importance in various engineering fields such as earthquake engineering, soil dynamics, aeronautics, nuclear reactors, etc. It is well known that the classical theory of thermoelasticity [1, 2] rests upon the hypothesis of the Fourier law of heat conduction, in which the temperature propagation is governed by a parabolic-type partial differential equation. The theory predicts that a thermal signal is felt instantaneously everywhere in a body. This is unrealistic from the physical point of view, especially for short-time responses. To account for the effect of thermal relaxation, generalized thermoelasticity has been formulated on the basis of a modified Fourier law such that the temperature propagation is governed by a hyperbolic-type equation. Accordingly, heat transport in solids is regarded as a wave phenomenon rather than a diffusion phenomenon.

In the paper [3], waves propagating along an arbitrary direction in a heat-conducting orthotropic thermoelastic plate are presented by utilizing the normal mode expansion method in the generalized theory of thermoelasticity with one thermal relaxation time. In the paper [4], the author studied the interaction of free harmonic waves with a multilayered medium in generalized thermo-elasticity by utilizing the combination of the linear transformation formation and transfer matrix method approach. Solutions obtained are general and pertain to several special cases. Of these mention the dispersion characteristics for a multilayered medium.

The wave propagation in an anisotropic inhomogeneous medium is considered. A new method of matricant has been developed. Based on the method matricant [5] treated wave processes in elastic and thermoelastic anisotropic media in anisotropic dielectric media, the waves in anisotropic plates, electromagnetic waves in media with magnetoelectric effect [6-8], the waves in liquid crystals, wave propagation in thermoelastic media [9-12].

The structure of matricant for the equation motion elastic medium equations, equations of thermo-mechanical medium has been established. Wave propagation in infinite and finite periodical inhomogeneous media are studied.

Research method

The research method is the matricant method [5] which allows to obtain accurate analytical solutions of differential equations describing the related processes in media with piezoelectric, piezomagnetic, thermoelastic and thermo-piezoelectric properties.

The method of study is analytical and is based on the development of matrix methods for studying the dynamics of elastic stratified media.

The method is about reducing the initial motion equations based on the variable separation method

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(representation of the solution in the form of plane waves) to the equivalent system of ordinary differential equations of the first order with variable coefficients and the construction of the matricant structure (normalized matrix of fundamental solutions).

In the case of the consistent approach, the matrix method allows considering the propagation of waves in a wide class of media. Another advantage of this method is that the expressions obtained by the matrix method have a very compact form, which proves to be convenient both for analytical studies and for numerical calculations.

The matrix method has been tested and the results obtained are consistent with previously known ones. This is confirmed by the presence of a large number of publications based on the above method.

Matrix formulation of the propagation of thermoelastic waves

Propagation of thermoelastic waves in anisotropic medium is described by the equations of motion to be solved together with the Fourier heat equation and the equation of heat flow, which have the form [1]:

$$\frac{\partial\sigma_{XX}}{\partial X}+\frac{\partial\sigma_{XY}}{\partial Y}+\frac{\partial\sigma_{XZ}}{\partial Z}=\rho\frac{\partial^2U_X}{\partial t^2}$$

$$\frac{\partial\sigma_{XY}}{\partial X}+\frac{\partial\sigma_{YY}}{\partial Y}+\frac{\partial\sigma_{YZ}}{\partial Z}=\rho\frac{\partial^2U_Y}{\partial t^2} \qquad (1)$$

$$\frac{\partial\sigma_{XZ}}{\partial X}+\frac{\partial\sigma_{YZ}}{\partial Y}+\frac{\partial\sigma_{ZZ}}{\partial Z}=\rho\frac{\partial^2U_Z}{\partial t^2}$$

$$\lambda_{ij}\frac{\partial\theta}{\partial x_j}=-q_i \qquad (2)$$

$$\frac{\partial q_i}{\partial x_i}=-i\omega\beta_{ij}\varepsilon_{ij}-i\omega\frac{c_\varepsilon}{T_0}\theta \qquad (3)$$

where σ_{ij} - stress tensor, ρ - density of the medium, λ_{ij} - thermal conductivity tensor, q_i - the vector of heat, ω - the angular frequency, β_{ij} - thermomechanical constants, $\beta_{ij}=\beta_{ji}$, ε_{ij} - the strain tensor, c_ε - specific heat at constant strain, $\theta = T-T_0$ - temperature increase compared with the temperature of the natural state T_0, $\left|\frac{\theta}{T_0}\right| \prec\prec 1$ for small deformations.

Physical and mechanical quantities are related by the relation of Duhamel-Neumann [2]:

$$\sigma_{ij}=c_{ijkl}\varepsilon_{kl}-\beta_{ij}\theta \qquad (4)$$

Here c_{ij} - elastic parameters, $c_{ijkl}=c_{jikl}=c_{ijlk}=c_{klij}$; ε_{kl} - the tensor Cauchy for small deformations.

For crystals of a hexagonal system as coordinate three orthogonal axes of symmetry or inversion axes of the second order get out.

For a hexagonal class of crystals, the ratio of Duhamel - Neumann looks like:

$$\begin{pmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{zz}\end{pmatrix}=\begin{pmatrix}c_{11}&c_{12}&c_{13}\\ c_{12}&c_{11}&c_{13}\\ c_{13}&c_{13}&c_{33}\end{pmatrix}\begin{pmatrix}\varepsilon_1\\ \varepsilon_2\\ \varepsilon_3\end{pmatrix}-\begin{pmatrix}\beta_{11}&0&0\\ 0&\beta_{11}&0\\ 0&0&\beta_{33}\end{pmatrix}\theta \qquad (4a)$$

$$\begin{pmatrix}\sigma_{yz}\\ \sigma_{xz}\\ \sigma_{xy}\end{pmatrix}=\begin{pmatrix}c_{44}&0&0\\ 0&c_{44}&0\\ 0&0&\frac{c_{11}-c_{12}}{2}\end{pmatrix}\begin{pmatrix}\varepsilon_4\\ \varepsilon_5\\ \varepsilon_6\end{pmatrix}$$

Equations (1), (2), (3), (4) and (4a) determine the relationship of mechanical stress and temperature as a function of the independent variables - the thermal field and deformation.

Thus, the relations (1) - (4) constitute a closed system of thermoelasticity equations, which describes the propagation of thermoelastic waves.

Based on the method of separation of variables in the case of a harmonic function of time [5]:

$$\left[U_i(x,y,z,t);\ \sigma_{ij}(x,y,z,t);\ \theta;\ q_z\right]=\left[U_i(x),\sigma_{ij}(x),\ \theta;\ q_z\right]e^{i(\omega t-ny-lz)} \qquad (5)$$

the system of equations (1) -- (4) reduces to a system of differential equations of first order with variable coefficients which describes the propagation of harmonic waves:

$$\frac{d\vec{W}}{dx}=B\vec{W} \qquad (6)$$

where: $\vec{W}$ is a column vector, $u_x(x)$, $u_y(x)$, $u_z(x)$ represent the projection of the displacement vector on the corresponding coordinates, and $m=k_x$, $n=k_y$, $l=k_z$, show the x, y and z components of a wave vector *k*, respectively; $\mathbf{B}=\mathbf{B}\lfloor c_{ijkl}(x),\beta_{ij}(x),\theta,\omega,m,n,l\rfloor$ - coefficient matrix which elements contain the parameters of the medium in which thermoelastic waves propagate.

The vector $\vec{W}$ has the form:

$$\vec{W}(x,y,z,t)=[u_x(x),\sigma_{xx},u_y(x),\sigma_{xy},u_z(x),\sigma_{xz},\theta,q_x]^t\exp(i\omega t-iny-ilz) \qquad (7)$$

The symbol t indicates the transpose of the vector - a vector of strings - column.

The system of differential equations (6) for a anisotropic medium of a hexagonal system looks like:

$$\frac{dU_x}{dx}=\frac{1}{c_{11}}\sigma_{xx}+\frac{c_{12}}{c_{11}}inU_y+\frac{c_{13}}{c_{11}}ilU_z+\frac{\beta_{11}}{c_{11}}\theta$$

$$\frac{d\sigma_{xx}}{dx} = -\rho\omega^2 U_x + in\sigma_{xy} + il\sigma_{xz}$$

$$\frac{dU_y}{dx} = \frac{2}{c_{11} - c_{12}}\sigma_{xy} + inU_x$$

$$\frac{d\sigma_{xy}}{dx} = in\,\frac{c_{12}}{c_{11}}\sigma_{xx} + \left[-\rho\omega^2 + n^2\left(c_{11} - \frac{c_{12}^2}{c_{11}}\right) + c_{44}l^2\right]U_y + nl\left(c_{13} - \frac{c_{12}c_{13}}{c_{11}} + c_{44}\right)U_z + \left(\frac{c_{12}}{c_{11}}\beta_{11} - \beta_{11}\right)in\,\theta$$

$$\frac{dU_z}{dx} = \frac{1}{c_{44}}\sigma_{xz} + ilU_x \qquad (6a)$$

$$\frac{d\sigma_{xz}}{dx} = il\,\frac{c_{13}}{c_{11}}\sigma_{xx} + nl\left[c_{44} - \frac{c_{13}c_{12}}{c_{11}} + c_{13}\right]U_y + \left(-\rho\omega^2 + n^2c_{44} - l^2\frac{c_{13}^2}{c_{11}} + c_{33}l^2\right)U_z + \left[\frac{c_{13}}{c_{11}}\beta_{11} - \beta_{33}\right]il\,\theta$$

$$\frac{d\theta}{dx} = -\frac{1}{\lambda_{11}}q_x$$

$$\frac{dq_z}{dx} = i\omega\frac{\beta_{11}}{c_{11}}\sigma_{xx} + i\omega n\left(\frac{c_{12}}{c_{11}} - 1\right)\beta_{11}U_y + i\omega l\left[\frac{c_{13}}{c_{11}}\beta_{11} - \beta_{33}\right]U_y + i\omega\left(\frac{\beta_{11}}{c_{11}} + \frac{c_\varepsilon}{T_0}\right)\theta$$

The heterogeneity of the medium is assumed along X. In constructing the coefficient matrix *B* is used as a representation of the solution (5), the system of equations (1) - (4) are in the derivatives along the coordinate X and the excluded components of the stress tensor are not included in the boundary conditions. The multiplier $\exp(i\omega t - iny - ilz)$ is omitted throughout.

In the structure of the matrix and vector - column boundary conditions in the bulk case for the hexagonal crystal system in the case of the symmetry axis of the second order and heterogeneity along the X axis are given by:

$$B = \begin{bmatrix} 0 & b_{12} & b_{13} & 0 & b_{15} & 0 & b_{17} & 0 \\ b_{21} & 0 & 0 & b_{24} & 0 & b_{26} & 0 & 0 \\ b_{24} & 0 & 0 & b_{34} & 0 & 0 & 0 & 0 \\ 0 & b_{13} & b_{43} & 0 & b_{45} & 0 & b_{47} & 0 \\ b_{26} & 0 & 0 & 0 & 0 & b_{56} & 0 & 0 \\ 0 & b_{15} & b_{45} & 0 & b_{65} & 0 & b_{67} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & b_{78} \\ 0 & i\omega b_{17} & i\omega b_{47} & 0 & i\omega b_{67} & 0 & b_{87} & 0 \end{bmatrix};$$

$$\vec{W} = \begin{pmatrix} u_x \\ \sigma_{xx} \\ u_y \\ \sigma_{xy} \\ u_z \\ \sigma_{xz} \\ \theta \\ q_x \end{pmatrix} \qquad (8)$$

From the structure of the coefficient matrix (8) that is in the spatial case, the elastic waves of different polarization and the heat wave are interrelated.

The b_{ij} elements of the coefficient matrix *B* for a hexagonal system in a volume case look like:

$$b_{12} = \frac{1}{c_{11}}; b_{13} = \frac{c_{12}}{c_{11}}in; b_{15} = \frac{c_{13}}{c_{11}}il; b_{17} = \frac{\beta_{11}}{c_{11}};$$

$$b_{21} = -\omega^2\rho;\;\; b_{24} = in;\;\; b_{26} = il$$

$$b_{34} = \frac{2}{c_{11} - c_{12}}; b_{43} = \left(c_{11} - \frac{c_{12}^2}{c_{11}}\right)n^2 + c_{44}l^2 - \omega^2\rho; b_{45} = \left(c_{44} + c_{13} - \frac{c_{12}c_{13}}{c_{11}}\right)nl;$$

$$b_{47} = \left(\frac{c_{12}}{c_{11}} - 1\right)\beta_{11}in$$

$$b_{56} = \frac{1}{c_{44}}; b_{65} = \left(c_{33} - \frac{c_{13}^2}{c_{11}}\right)l^2 + c_{44}n^2 - \omega^2\rho; b_{67} = \left(\frac{c_{13}}{c_{11}}\beta_{11} - \beta_{33})\right)il;$$

$$b_{87} = -i\omega\left(-\frac{\beta_{11}}{c_{11}} + \frac{c_\varepsilon}{T_0}\right); b_{78} = -\frac{1}{\lambda_{11}}$$

The nonzero elements of the matrix of coefficients *B* b_{13}, b_{24} determine the mutual transformation of longitudinal and transverse X - polarized waves. Elements of b_{15}, b_{26} describe the relationship of transverse X-polarization with the longitudinal wave. Nonzero element b_{45} defines the mutual transformation between the waves of transverse polarization.

The fact that the coefficient b_{17}:

$$b_{17} = \frac{\beta_{11}}{c_{11}}$$

means that the longitudinal wave is propagated by the thermoelastic effect.

Non-zero elements b_{47} and b_{67}:

$$b_{47} = \left(\frac{c_{12}}{c_{11}} - 1\right)\beta_{11}in\,; b_{67} = \left(\frac{c_{13}}{c_{11}}\beta_{11} - \beta_{33}\right)il;$$

indicate the effect on the elastic wave transverse polarizations thermoelastic effect. At the same time. it describes the b_{47} thermoelastic effect on the elastic shear wave of the Y-polarization, and the b_{67} thermoelastic effect on the transverse wave Z-polarization.

Similarly, for the thermoelastic waves propagating in an anisotropic medium of hexagonal symmetry the coefficient matrix is constructed in the bulk case and the analysis of matrix coefficients is peerformed. We also obtained that the structure of the matrix of coefficients in the propagation of thermoelastic waves in an anisotropic medium of hexagonal crystal systems in the planes XY and XZ, defines the types of waves and the mutual transformation of waves of different polarizations.

Piezoelastic waves

The existence of direct and reverse piezoelectric effects in a dielectric medium leads to the mutual generation of elastic and electromagnetic waves. A complete description of the processes of propagation of elastic and electromagnetic waves is based on the analysis of joint solutions of the equations of motion of an elastic anisotropic medium (1) and Maxwell's equations [13]. Electromagnetic wave processes are considered on the basis of Maxwell's equations in the absence of free charges and currents in the medium:

$$\begin{cases} rot\,\vec{\mathbf{E}} = -\dfrac{\partial \vec{\mathbf{B}}}{\partial t} \\ rot\,\vec{\mathbf{H}} = \dfrac{\partial \vec{\mathbf{D}}}{\partial t} \end{cases} \quad \begin{cases} div\,\vec{\mathbf{B}} = 0 \\ div\,\vec{\mathbf{D}} = 0 \end{cases} \tag{9}$$

$$D_i = \epsilon_{ij} E_i; \quad B_i = \mu_{ij} H_i$$

where: E_i are the components of the electric field strength vector, D_i are the components of the electric displacement vector, B_i are the components of the magnetic induction vector, and H_i are the components of the magnetic field strength vector. The coefficients ϵ_{ij}, μ_{ij} are dielectric and magnetic parameters of the medium, which included μ_0, ε_0 are the magnetic and dielectric permeability of free space, respectively.

System of Eqs. (1), (9) coupled with material equations:

$$\begin{cases} \sigma_{ij} = c_{ijkl}\varepsilon_{kl} - e_{ijk}E_k \\ D_i = \epsilon_{ik} E_k + e_{ikl}\varepsilon_{kl} \end{cases} \tag{10}$$

where e_{ikl} is the piezoelectric tensor, which determines the interaction of elastic and electromagnetic fields and can be represented as a (3×6) matrix. Application of the representation of solutions for the desired function in the form (5) reduces the system of Eqs. (9) and equations of motion (1) with (10) to a system of first-order equations [14]:

$$\frac{d\vec{\mathbf{W}}}{dz} = \mathbf{B}\vec{\mathbf{W}};$$

$$\vec{\mathbf{W}} = (u_z, \sigma_{zz}, u_x, \sigma_{xz}, u_y, \sigma_{yz}, E_y, H_x, H_y, E_x)^t \tag{11}$$

As a result, the matrix of coefficients **B** has (10×10) order. For example, when we consider waves in orthorhombic media of class *222*, when the projection of the wave vector k_y=0 (the plane xz), the matrix is divided into a (6×6) and a (4×4) matrix. The (4×4) matrix describes propagation of coupled shear elastic waves with Y-polarization and TM electromagnetic waves. The structure of the matrix of coefficients ***B*** has the form [14]:

$$\mathbf{B} = \begin{pmatrix} 0 & b_{12} & 0 & b_{14} \\ b_{21} & 0 & b_{23} & 0 \\ 0 & -i\omega b_{14} & 0 & b_{34} \\ -i\omega b_{23} & 0 & b_{43} & 0 \end{pmatrix};$$

$$\vec{\mathbf{W}} = (u_y, \sigma_{yz}, H_y, E_x)^t$$

(12)

where: $b_{21} = k_x^2(C_{66} + \frac{e_{36}^2}{\epsilon_{33}}) - \rho\omega^2$;

$b_{12} = \frac{1}{C_{44}}$;

$b_{23} = \frac{im^2 e_{36}}{\omega\epsilon_{33}}$;

$b_{14} = \frac{e_{14}}{C_{44}}$;

$b_{34} = -i\omega(\epsilon_{11} + \frac{e_{14}^2}{C_{44}})$;

$b_{43} = i\omega(\frac{m^2}{\omega^2\epsilon_{33}} - \mu_{22})$.

Piezomagnetic media

Previously, the paper [15] considered physical models describing piezomagnetic media coupled by elastic and electromagnetic fields, based on the system of equations (1), (9) in combination with material equations for piezomagnetic media:

$$\begin{aligned} \sigma_{ij} &= c_{ijkl}\varepsilon_{kl} - Q_{ijk}H_k \\ B_i &= \mu_{ik}H_k + Q_{ikl}\varepsilon_{kl} \end{aligned} \tag{13}$$

where Q_{ijk} are piezomagnetic modules of the anisotropic media. Application of the representation solution (5) allows Eqs. (1), (9), and (12) to give a system of first-order ODEs:

$$\frac{d\vec{\mathbf{W}}}{dz} = \mathbf{B}\vec{\mathbf{W}};$$

$$\vec{\mathbf{W}} = (u_z, \sigma_{zz}, u_y, \sigma_{yz}, u_x, \sigma_{xz}, E_y, H_x, H_y, E_x)^t \tag{14}$$

The matrix of coefficients **B** in the general case has order (10×10) The structure of this matrix coefficients is obtained for orthorhombic media [15].

Magnetoelectric media

In the work [16], Maxwell's system of equations and constitutive equations describing the propagation of electromagnetic waves in an anisotropic magnetoelectric medium are equated to an equivalent system of differential equations of first order. This gives an opportunity to analyze magnetoelectric effect on electromagnetic wave propagation along axes planes and in bulk case.

Under absence of volume charge density, current density vectors and harmonic dependence of the wave fields solutions on time Maxwell's equations take following form:

$$rot\vec{H} = \frac{\partial \vec{D}}{\partial t} = i\omega\vec{D}; \tag{15}$$

$div\vec{B} = 0; \qquad div\vec{D} = 0;$

The dependence of $\vec{D}$ and $\vec{B}$ on $\vec{E}$ and $\vec{H}$ in presence of magnetoelectric effect has the following form:

$$\begin{cases} D_i = \varepsilon_0 \varepsilon_{ij} E_j - \alpha_{ij} H_j \\ B_i = \mu_0 \mu_{ij} H_j - \alpha_{ij} E_j \end{cases} \qquad (16)$$

where: ε_0, μ_0 - absolute dielectric permeability of vacuum; ε_{ij}, μ_{ij} - components of relative dielectric and magnetic permeability of medium. α_{ij} - components of the tensor that describe the influence of the magnetoelectric effect.

In general, the matrix of B coefficients has the following structure:

$$B = \begin{pmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{11} & b_{23} & b_{24} \\ -b_{24} & -b_{14} & -b_{11} & b_{34} \\ -b_{23} & -b_{13} & b_{43} & -b_{11} \end{pmatrix} \qquad (17)$$

For the antiferromagnetic Cr_2O_3 that is being considered in this article tensor $\hat{\alpha}$ has the following form:

$$\hat{\alpha} = \begin{pmatrix} \alpha_{II} & 0 & 0 \\ 0 & \alpha_{II} & 0 \\ 0 & 0 & \alpha_{\perp} \end{pmatrix} \qquad (18)$$

CONCLUSION

This paper is devoted to the research of thermoelastic wave propagation in anisotropic media of hexagonal systems in the case of a second-order axis symmetry and heterogeneity along the X axis. Differential equations system of the first order with variable coefficients that are made by means of the variable separation method are obtained (solution is presented as a plane harmonic wave). Coefficients matrices for anisotropic medium of a hexagonal system for three-, two-, and one-dimensional cases were obtained. The structures of the matrices of the coefficients of the constitutive equations and the structure of the matrix for waves of an elastic and electromagnetic coupled field in thermoelastic, piezoelectric, piezomagnetic and magnetoelectric anisotropic media are presented.

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